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FUNDAMENTALS OF THE STATISTICAL THEORY OF FRACTURE,(U)

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The first important study of fracture statistics was that of Weibull. His work was based on the tacit assumption that only the component of stress normal to a crack plane contributes to its fracture, and on the use of simple analytical formulas for failure probability. Recent progress in short-term fracture includes the use of more refined fracture criteria and a search for better distribution functions for the frequency of cracks, based on microstructural considerations. Use of the critical value of strain energy

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release rate as a fracture criterion leads to improved agreement with experiment. Consideration is also given to the statistics of fracture in static fatigue and in dynamic fracture.

FUNDAMENTALS OF THE STATISTICAL THEORY OF FRACTURE

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ABSTRACT

The first important study of fracture statistics was that of Weibull. His work was based on the tacit assumption that only the component of stress normal to a crack plane contributes to its fracture, and on the use of simple analytical formulas for failure probability. Recent progress in short-term fracture includes the use of more refined fracture criteria and a search for better distribution functions for the frequency of cracks, based on microstructural considerations. Use of the critical value of strain energy release rate as a fracture criterion leads to improved agreement with experiment. Consideration is also given to the statistics of fracture in static fatigue and in dynamic fracture.

INTRODUCTION

Ceramic materials are generally brittle, and nominally identical specimens of a brittle material may exhibit large variations in fracture stress, especially if the specimens are small. When brittle materials are employed in practical structures, the designer must be able to assure himself of an acceptably low probability of failure during service. Typically, on the basis of laboratory data on a limited number of specimens uniformly loaded in simple tension or pure bending, the probability of failure must be calculated for structural members of different sizes and shapes and under completely different loading conditions. The tool for accomplishing this is fracture statistics.

The first important contribution to this subject was made by Weibull in 1939 [1,2], and his theories are still the basis for most calculations in this field. He exploited the analogy between a stressed brittle structure and a loaded chain, which breaks when the strength of its weakest link is exceeded. Before discussing weakest link theory [WLT], it is appropriate to enquire into its range of applicability.

Clearly, the fracture of a single fiber is similar to the breaking of a chain. The fiber may be regarded as a linear array of very short elements, and the fracture of any element causes failure of the fiber as a whole. Similarly, when an isolated crack in an elastic body is loaded normal to its plane, it will become unstable and grow catastrophically, causing fracture. The fracture stress of the entire body is that of the weakest crack. On the other hand, there are many situations to which WLT does not apply. For instance, in a bundle of fibers the first fiber failure does not ordinarily cause failure in the bundle as a whole; rather, failure is the result of damage accumulation. Similarly, in some stress states (e.g. pure compression) a crack in an elastic body will generally grow in a direction which results in crack arrest. Here, also, failure is a result of progressive damage, and WLT does not apply.

Ideally, fracture statistics should be based on a proper consideration of three elements — extreme value statistics, fracture mechanics, and material microstructure. Weibull's theory is based almost exclusively on the first element. In recent years, progress has been made in incorporating the other two elements into weakest link theory. The purpose of the present paper is to outline the current status of fracture statistics and indicate some directions of expected future progress. In the effort to give a clear and concise description of the field, historical perspective has been somewhat slighted. The primary emphasis is on the underlying physical concepts rather than computational techniques. It is hoped that those whose contributions may have been omitted or underemphasized will understand and forgive.

WEAKEST LINK THEORY (WLT)

We first derive the fundamental equation of weakest link theory. Let it be assumed that a stressed solid can fail due to any of a number of independent and mutually exclusive mechanisms or causes, each involving infinitesimal probability of failure $(\Delta P_f)_i$. The probability that the i'th mechanism will not cause failure is $(P_S)_i = 1 - (\Delta P_f)_i$. The overall probability of survival is the product of the individual probabilities of survival, i.e.,

$$P_S = \prod_i \left(P_S\right)_i = \prod_i \left\{1 - \left(\Delta P_f\right)_i\right\}$$

$$\cong \prod_i \exp\left[-\left(\Delta P_f\right)_i\right] = \exp\left[-\sum_i \left(\Delta P_f\right)_i\right] \quad . \tag{1}$$

The sum of the individual probabilities of failure appearing in the final equality above was called by Weibull the "risk of rupture" and was given the symbol B.

In evaluating B, Weibull added together the probabilities of failure of all the elements of volume ΔV in the entire body. The probability of failure of the i'th element ΔV_i in simple tension σ, for example, is

$$\left(\Delta P_f\right)_i = n(\sigma)\ \Delta V_i \quad , \tag{2}$$

where $n(\sigma)$ is the number of flaws per unit volume with a strength less than σ. If $n(\sigma)$ is less than unity, it can be regarded as the probability that such a flaw will occur in a unit volume. Thus, the probability of failure is given by

$$P_f = 1 - P_S = 1 - \exp\left[-\int dV\ n(\sigma)\right] \quad . \tag{3}$$

For a uniformly stressed body

$$P_f(\sigma) = 1 - \exp\left[-V\ n(\sigma)\right] \quad , \tag{4}$$

or

$$n(\sigma) = \frac{1}{V} \ln P_S^{-1} \quad . \tag{5}$$

If $P_f(\sigma)$ is determined by testing a number of specimens in simple tension, Eq. (5) can be used to obtain $n(\sigma)$ and Eq. (3) can then be used to evaluate the probability of failure of a body of arbitrary size and shape and nonuniformly distributed tensile stress. By assuming, as Weibull did, that compressive stresses do not contribute to fracture, the above procedure can also be used to calculate the probability of failure in pure bending, or indeed any combination of uniaxial tension and compression. A limitation to this procedure is that in ignoring fracture due to compression, it does not adequately account for failure in all stress states.

Weibull also gave a procedure for calculating P_f for polyaxial stress states when the failure statistics for simple tension are known. Basically, his procedure is to calculate B by averaging the tensile stress in all directions [1,3]. This is intuitively plausible, but not rigorous, and it has been questioned by some investigators. Barnett et al. [4] have formulated a number of alternative procedures they considered equally plausible. Presumably, in part at least, because of doubts concerning Weibull's procedure, some investigators [5,6,7] have chosen to use an approximation in which it is assumed that with respect to fracture, the principal stresses act independently, i.e.,

$$P_S(\sigma_1, \sigma_2, \sigma_3) = P_S(\sigma_1)\, P_S(\sigma_2)\, P_S(\sigma_3) \quad , \tag{6}$$

where σ_1, σ_2, σ_3 are the principal stresses.

There is some evidence that Weibull had second thoughts concerning the validity of his recommended procedure, since he wrote in 1966 [8]: "Another problem of a more theoretical nature will be to deduce the effect of bi- and triaxiality on the distribution functions of one-dimensional stresses. If the principal stresses are acting independently of each other, and it seems that such materials may exist, then the problem may be soluble along lines previously sketched. In other cases the solution is very intricate and will certainly require close examination of the physical behavior of the material in question."

WLT FOR POLYAXIAL STRESS CONDITIONS

To put weakest link theory for polyaxial stress states on a firm physical foundation, we will make the explicit assumption that the flaws responsible for fracture are microcracks in the material. We will further assume that the cracks do not interact, and that each crack has a critical stress σ_c defined as the remote tensile stress applied normal to the crack plane which will cause fracture. Fracture under combined stresses occurs when the effective stress σ_e acting on the crack is equal to σ_c. The effective stress is some function of the applied stresses at the location of the crack, and its precise form depends on the fracture criterion employed. The search for fracture criteria leads us to fracture mechanics. Before discussing particular fracture criteria, however, we formulate the theory of fracture statistics for an arbitrary fracture criterion by leaving σ_e unspecified.

The potential causes of failure are the individual cracks. For purposes of analysis, it is convenient to group the cracks according to location, the applied stress state, and crack critical stress. We assume that the stress state varies slowly so that within a volume element ΔV all cracks will be subject to the same macroscopic stress. We also assume that the material is macroscopically homogeneous, so that a function $N(\sigma_c)$ can be defined as the number of cracks per unit volume having a critical stress equal to or less than σ_c. The probability that a crack having a critical stress in the range σ_c to $\sigma_c + d\sigma_c$ exists in volume element ΔV is the $\Delta V\ [dN(\sigma_c)/d\sigma_c]\ d\sigma_c$.

If such a crack is actually present, the probability that it will fracture depends on its orientation, the stress state, and the fracture criterion. We assume there is a solid angle Ω such that fracture of a crack will occur if and only if its normal lies within Ω. This means that if the normal lies within Ω, $\sigma_e > \sigma_c$, where σ_e is the effective stress corresponding to the fracture criterion selected. If the cracks are randomly oriented, the probability that a crack will fracture under the applied stress Σ is $\Omega(\Sigma, \sigma_c)/4\pi$.

Now the probability of failure due to a crack in the critical stress range $d\sigma_c$ located in volume element ΔV is the product of the above probabilities, i.e.,

$$\left(\Delta P_f\right)_i = \left(\Delta V \frac{dN(\sigma_c)}{d\sigma_c}\ d\sigma_c\right)\left(\frac{\Omega(\Sigma, \sigma_c)}{4\pi}\right) \quad . \tag{7}$$

Substituting Eq. (7) into Eq. (1), and changing sums into integrals, we obtain

$$P_S = \exp\left[-\int dV \int d\sigma_c \frac{dN}{d\sigma_c} \frac{\Omega}{4\pi}\right] \quad . \tag{8}$$

We note in passing that $N(\sigma_c)$ is independent of stress state and depends only on the material. Also, since σ_c is defined as the stress which causes fracture when applied normal to the crack plane, $N(\sigma_c)$ can be converted directly into crack size distribution when K_{Ic} is known. In these respects it differs from Weibull's n and the g(S) of McClintock and others [9,10,11]. These functions represent the number of cracks per unit volume that will be fractured by a particular applied stress state, and derivative of such a function respectively, and they depend on the stress ratio. For the particular case of hydrostatic tension, $N(\sigma) = n(\sigma)$ and $N'(\sigma) = g(\sigma)$, because then $\Omega = 4\pi$ or zero depending on the stress level.

In simple tension and equibiaxial tension, analytical expressions can be found for Ω for some fracture criteria, at least, and used together with Eq. (3) to evaluate P_S. In the general case, we find Ω by integrating $d\Omega$ over the range in which $\sigma_e > \sigma_c$. One way of accomplishing this is to integrate over the entire angular range but include a suitable operator H in the integral:

$$P_S = \exp\left[-\iiint dV\, d\Omega\, d\sigma_c\, H(\sigma_e,\sigma_c)\, \frac{dN}{d\sigma_c}\right] \quad , \tag{9}$$

where

$$\begin{aligned} H(\sigma_e,\sigma_c) &= 1 \text{ when } \sigma_e > \sigma_c \\ &= 0 \text{ where } \sigma_e < \sigma_c \quad . \end{aligned} \tag{10}$$

n now carry out the integral over σ_c first, with the result

$$P_S = \exp\left[-\iint dV\, d\Omega\, N(\sigma_e)\right] \quad . \tag{11}$$

The effective stress causing fracture is a function of both the component of stress normal to the crack plane, σ_n, and the shear stress τ parallel to the crack plane. An approximation frequently employed is the assumption that the cracks are shear-insensitive, i.e., that $\sigma_e = \sigma_n$. Using this approximation, Eq. (11) becomes equivalent to Weibull's rule for polyaxial stress states [1], and Eq. (8) reduces to the equation of Batdorf and Crose [12]. Eq. (8) and Eq. (11) are equivalent formulations of the same theory. The physical justification of the former is the more readily apparent, but the latter is more convenient for computational purposes.

RELATION TO GRIFFITH THEORY

In 1924, Griffith [13] published a theory for the fracture of solids under biaxial stress conditions. He assumed the presence of a large number of identical cavities in the form of nearly flat elliptical cylinders (Griffith cracks) with their axes normal to the plane of the stresses, but otherwise randomly oriented. He calculated the maximum peripheral tensile stress on the free surface of such a crack as a function of crack orientation, and assumed fracture would occur whenever this maximum value exceeded the intrinsic strength of the material.

This fracture criterion led to the failure envelopes shown in Fig. 1. In the tension-tension quadrant and most of the tension-compression quadrant, failure obeys a maximum tensile stress criterion. Fracture initiates in a crack whose plane is normal to the largest principal tensile stress. Thus, only the normal stress σ_n acts on the crack plane, and tensile-type fractures result. In the portion of the tension-compression quadrant near the compression axis, material strength is locally exceeded when the crack normal is inclined at somewhat more than 45° to the axis of maximum

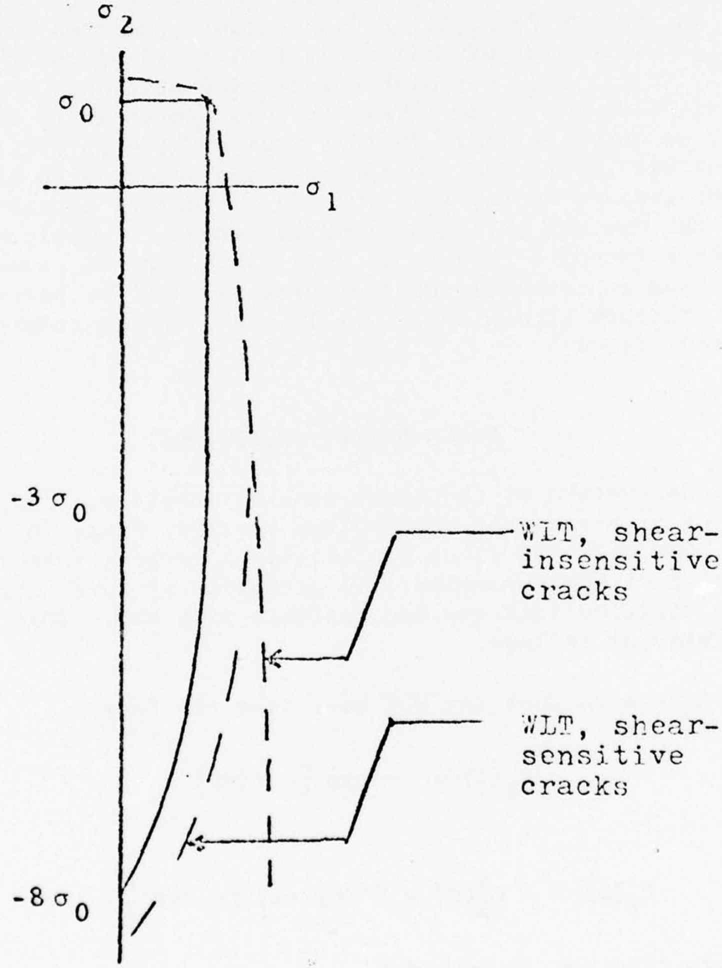

Fig. 1. Comparison of Fracture Theories for Biaxial Stresses

tension. Here both σ_n and τ contribute to failure and the resulting oblique fracture is often called a shear-type failure.

In any crack-based statistical theory it is most unlikely that the weakest crack will be oriented at exactly the angle of maximum vulnerability, so Griffith's curves must be regarded as the lower limit for fracture, i.e., they represent the contour for $P_f(\sigma_1,\sigma_2) \lll 1$. A typical $P_f(\sigma_1,\sigma_2) = 0.5$ contour, for the case of shear-insensitive cracks, is shown as a dashed curve. This curve covers both quadrants almost completely, but does not provide for failure due to pure compression. In the case of shear-sensitive cracks, the entire $P_f(\sigma_1,\sigma_2) = 0.5$ contour predicted by WLT will fall slightly outside of Griffith's failure envelope as shown schematically in Fig. 1. For high compressive stresses, the results differ significantly from those for shear-insensitive cracks. However, as noted earlier, in this region WLT does not apply. Thus, Weibull theory and the more refined theories to be discussed later are limited to cases in which no principal compressive stress exceeds the maximum principal tensile stress in absolute value by more than a factor of about 3. This limitation is probably not too serious from a practical point of view because the percent dispersion in fracture stress is generally much less in compression than in tension [14,15].

CRACK DENSITY FUNCTIONS

The derivative of the crack density function, $N'(\sigma_c)$, gives the number of cracks per unit volume per unit range in critical stress. One might at first be inclined to expect this function to be Gaussian or near-Gaussian. In actuality it turns out that Gaussian distributions are incompatible with WLT. This is readily demonstrated as follows.

We have seen that any WLT must take the form

$$P_S(\sigma) = - \exp\left[- f(\sigma)\right] \quad , \tag{12}$$

whence

$$P_f'(\sigma) = - P_S'(\sigma) = f'(\sigma) \exp\left[- f(\sigma)\right] \quad . \tag{13}$$

If the distribution is Gaussian,

$$P_f'(\sigma) = A \exp\left[- a (\sigma - b)^2\right] \quad . \tag{14}$$

Eq. (13) and Eq. (14) are incompatible, because if $f'(\sigma)$ is a constant, $f(\sigma)$ cannot be a quadratic function. Another reason for excluding the Gaussian distribution is that it implies a nonvanishing probability of tensile failure when the applied stress is compression.

Weibull introduced two distribution functions

$$P_f(\sigma) = 1 - \exp\left[- V k\sigma^m\right] \tag{15}$$

and

$$P_f(\sigma) = 1 - \exp\left[- Vk (\sigma - \sigma_u)^m\right] \quad (\sigma > \sigma_u) \tag{16a}$$

$$= 0 \qquad\qquad (\sigma < \sigma_u) \tag{16b}$$

Each may be regarded as a skewed Gaussian distribution with a skewness, which can be either positive or negative, determined by the value of m. The 2-parameter form allows failure to occur at any positive value of the tensile stress, while the 3-parameter form implies that fracture cannot occur for $\sigma < \sigma_u$.

The function $P_f(\sigma)$ is conventionally determined by conducting N tests and numbering the observed fracture stresses $\sigma_1 \ldots \sigma_N$ in ascending order. It is then usually assumed that

$$P_f(\sigma_j) = \frac{j}{N + 1} \quad . \tag{17}$$

A more sophisticated statistical treatment [16] leads to the conclusion that

$$P_f(\sigma_j) = \frac{j - 0.3}{N + 0.4} \quad . \tag{18}$$

The difference between Eq. (17) and Eq. (18) becomes insignificant for large values of N and for simplicity we shall use Eq. (17).

A simple technique for determining the parameters m, σ_u, and k is to write Eq. (16a) in the form

$$\ln \ln (1 - P_f)^{-1} = \ln Vk + m \ln (\sigma - \sigma_u) \quad . \tag{19}$$

Next, $\ln \ln (1 - P_f)^{-1}$ is plotted against $\ln (\sigma - \sigma_u)$ for various assumed values of σ_u. The value of σ_u adopted is that for which the N data points most nearly lie on a straight line. The slope of the line chosen is m, and Vk is the value of $\ell n\ (1 - P_f)^{-1}$ for $\sigma = \sigma_u + 1$. When Weibull's 2-parameter form is used, σ_u is arbitrarily taken to be zero.

Before passing on, we note that a least squares fit in $\ell n\ \ell n\ (1 - P_f)^{-1}$ vs. $\ell n\ (\sigma - \sigma_u)$ space is in general not a least squares fit in P_f vs. σ space. Consequently, the Weibull form found in the manner described is not the best possible fit to the data. However, if the scatter in the data is not large, it will be adequate for most practical purposes.

Fracture statistics are relatively simple when Weibull's 2-parameter form is used. If the probability of failure in simple tension is

$$P_f = 1 - \exp\ (- V\ k_T\ \sigma^m) \quad , \tag{20}$$

then that in bending can be shown to be [1]

$$P_f = 1 - \exp\ [- V\ k_B\ \sigma^m)] \quad , \tag{21}$$

where k_B/k_T depends on the cross section. For a rectangular cross section

$$k_B = k_T/(2m + 2) \quad . \tag{22}$$

Under the uniform principal stresses σ_1, σ_2, σ_3, the probability of failure is given by

$$P_f = 1 - \exp\left[- V\ k \left(\frac{\sigma_2}{\sigma_1}, \frac{\sigma_3}{\sigma_1}\right) \sigma_1{}^m\right] \quad . \tag{23}$$

The constant k $(\sigma_2/\sigma_1, \sigma_3/\sigma_1)$ is an analytical function of the stress ratios [17]. Many other cases are treated in [18].

Weibull's 3-parameter representation has a similar universal applicability when attention is limited to uniaxial stress problems. In going from uniaxial to polyaxial stress states, however, the functional form changes [19], i.e. it is not possible to express both uniaxial and polyaxial stress data in the form $P_f = 1 - \exp\ [- V\ k\ (\sigma - \sigma_u)^m]$. To assist those desiring to use Weibull's 3-parameter representation in conjunction with his theory for polyaxial stress states, Dukes [20] has carried out parametric calculations with the aid of a computer.

Weakest link theory implies a volume effect that is sometimes misapplied. Consider, for instance, the situation illustrated schematically in Fig. 2. Let us assume specimens were tested, so 9 we know $P_f(\sigma)$ over the range $0.1 \lesssim P_f \lesssim 0.9$. The known region is shown as a solid curve, and the dotted extension is an extrapolation of the data, accomplished by assuming that the mathematical function used to represent the test data continues to be valid outside the range in which it was tested. All that WLT can tell us is that a specimen 10 times larger will have 10 times the risk of rupture over the stress range tested. It does not answer the generally more interesting question, what stresses correspond to

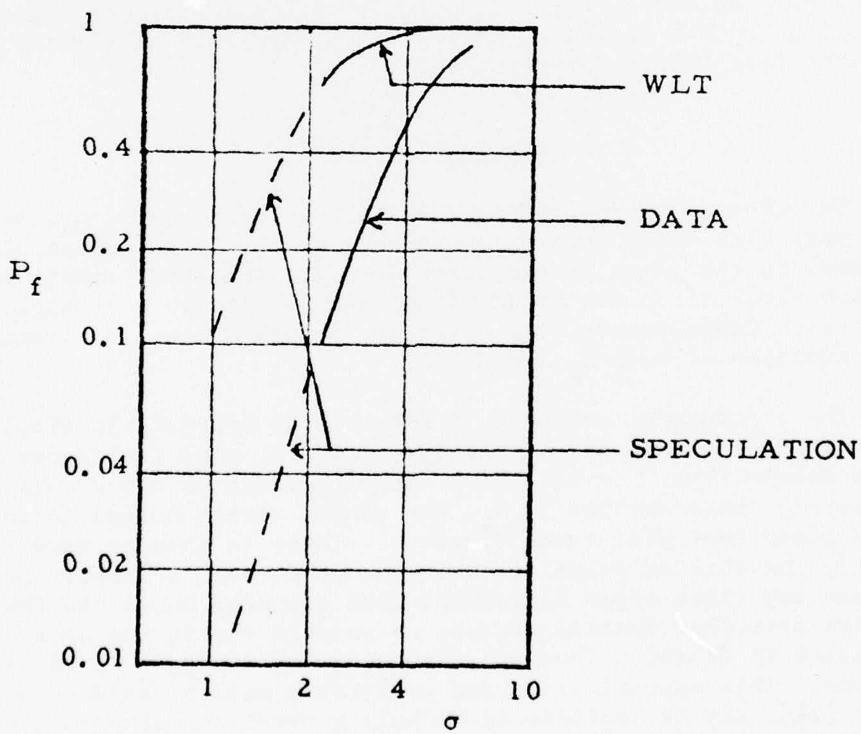

Fig. 2. Data, WLT inference concerning volume effect, and speculative extrapolations.

probabilities of failure of the larger specimen in the probability range tested, i.e., $0.1 \lesssim P_f \lesssim 0.9$, unless the validity of the extrapolation to lower stress levels can be established.

This raises the question whether the Weibull distribution functions have a fundamental significance that might justify confidence in such extrapolation. Weibull made no such claim; he regarded them only as convenient mathematical tools having wide applicability [21]. It is true that they coincide with the third asymptotic form of extreme value theory. However, the founders of this theory did not assert that all distributions must approach a limiting form, but only that if a distribution does approach a limiting form, it must be one of the three identified in the theory of extreme values [22,23]. Basically, the limiting form results when a distribution can be represented as a power series. When this happens, only the leading term need be considered for large samples (i.e. large specimens with very many cracks). We shall show later, however, that microstructural considerations suggest that such a power series is not possible, and that as a result no limiting form exists for fracture.

FRACTURE CRITERIA

The direct stress applied normal to a crack plane, σ_n, results in a very high local stress at the root of the crack, while direct stresses in the plane of the crack do not. The shear stress τ applied parallel to the crack plane also results in very high local stresses. Consequently, the effective stress σ_e causing fracture is a function of both σ_n and τ.

The statistical analysis of fracture is appreciably simplified, however, by the assumption that $\sigma_e = \sigma_n$, i.e. that the cracks are shear-insensitive. In this case the properties of the cracks are completely characterized by σ_c, the remote stress normal to the crack plane that will cause fracture. There is then no need to specify the size or shape of the crack or Poisson's ratio. And because any given crack is weakest when oriented normal to the largest principal tensile stress, as pointed out in the section "Relation to Griffith Theory," the approximation $\sigma_e = \sigma_n$ is not a bad one. This approximation was explicitly made by Batdorf and Crose [24], and is implicit in Weibull's treatment of polyaxial stress states [1].

The fracture criterion used by Griffith to take into account the effects of shear is the assumption that fracture occurs when the local tensile stress at some point on the crack surface exceeds the intrinsic strength of the material. Many authors have discussed the stress distribution around cavities of various types under various loading conditions. For present purposes, the most concise and general treatment is probably that of Mirandy and Paul [25] who give an explicit formula for the maximum stress on the surface of an ellipsoid with semiaxes a $\geq$ b >> c under arbitrary applied loads. They find that

$$\sigma_{max} = \frac{b}{c}\ \frac{1}{E}\left\{\sigma_n + \sqrt{\sigma_n^2 + \tau^2\ F}\right\} \quad , \qquad (24)$$

where E is an elliptic integral that depends on b/a and F is a complicated function of the geometry of the crack and its orientation relative to the shear stress. We have defined the effective stress to be σ_n in the absence of shear, and some function of σ_n and τ when shear is present. Accordingly, we conclude that for a material containing cracks of only a single plan form (i.e. a fixed value of b/a)

$$\sigma_e = \frac{1}{2}\left\{\sigma_n + \sqrt{\sigma_n^2 + \tau^2\ F}\right\} \quad . \qquad (25)$$

Using the formulas in [25], it can be shown that for a Griffith crack (G.C.) with its axis normal to the applied shear stress

$$\sigma_e = \frac{1}{2}\left\{\sigma_n + \sqrt{\sigma_n^2 + \tau^2}\right\} \quad (G.C.) \quad , \qquad (26)$$

while for a penny-shaped crack (P.S.C.)

$$\sigma_e = \frac{1}{2}\left\{\sigma_n + \sqrt{\sigma_n^2 + \tau^2/(1 - 0.5\nu)^2}\right\} (P.S.C.) \quad , \qquad (27)$$

where ν is Poisson's ratio. The latter, possibly less familiar result, is also given in [26].

There are, however, grounds for doubting that maximum local tensile stress represents an acceptable fracture criterion. Consider, for instance, an ellipsoidal cavity with principal axes a = 4b loaded in tension parallel to the c axis. The maximum local stress occurs in the equatorial plane, and it is uniform around the entire circumference. Thus, according to the maximum local tensile stress criterion, all points on the equatorial belt are equally likely to fracture and the crack should advance in all directions in its own plane. On the other hand, the region which is above some fixed fraction of the maximum stress is four times as wide at the ends of the b axis as at the ends of the a axis. Thus, if stressed volume is important, growth should start first at the ends of the b axis. Similarly, the stress intensity factor is largest at the ends of the b axis, where it is twice as large as at the ends of the a axis. If stress intensity governs fracture, the b axis should extend first, and the crack should circularize. It is generally agreed that this is what actually happens.

Another fracture criterion is the assumption that fracture occurs when the elastic strain energy released when the crack grows is equal to the energy stored in the newly created free surfaces, i.e. when the strain energy release rate reaches some critical value. The strain energy release rates are well-established for a Griffith crack or penny-shaped crack growing in its own plane. For the former [27]

$$G = \frac{1 - \nu^2}{E}\left[K_I^2 + K_{II}^2\right] \quad , \tag{28}$$

where

E = elastic modulus

$$K_I = \sigma\sqrt{\pi a} \tag{29}$$

$$K_{II} = \tau\sqrt{\pi a} \quad . \tag{30}$$

Thus,

$$\sigma_e = \sqrt{\sigma_n^2 + \tau^2} \quad \text{(G.C.)} \quad . \tag{31}$$

For the latter [28]

$$G = \frac{(1 - \nu^2)}{E}\left[K_I^2 + K_{II}^2 + \left(\frac{1}{1 - \nu^2}\right) K_{III}^2\right] \quad , \tag{32}$$

where

$$K_I = 2\sigma_n \sqrt{a/\pi} \tag{33}$$

$$K_{II} = 4\ \tau\ \sqrt{a/\pi}\,\sin\gamma/(2 - \nu) \tag{34}$$

$$K_{III} = 4\ \tau\ \sqrt{a/\pi}\,\cos\gamma\,(1 - \nu)/(2 - \nu) \quad . \tag{35}$$

The minimum value of G occurs at the points on the crack periphery where only modes I and II are involved, and here

$$\sigma_e = \sqrt{\sigma_n^2 + \tau^2/(1 - 0.5\nu)^2} \quad \text{(P.S.C.)} \quad . \tag{36}$$

Unfortunately, under the combined action of σ_n and τ, cracks do not extend in their own plane so the above expressions are not strictly correct. There is at present no consensus regarding the proper fracture criterion for cracks extending out of their plane. Swedlow [29] listed nearly thirty differing treatments of this subject that had appeared by 1975, and others have been proposed since that time. We will not attempt to employ any of these theories in this review. Instead, we will use the equations just derived noting, however, that the true value of σ_e must be smaller than that found for coplanar extension. That is because the lowest instability mode is the mode that actually occurs.

Some idea of the differences between the various fracture criteria discussed herein can be obtained by assuming the material obeys the Weibull 2-parameter form [Eq. (15)], and plotting the ratio of the risk of rupture under equibiaxial tension to that for uniaxial tension vs. the parameter m. This is done in Fig. 3 which is adapted from [17] and [30]. The ratio is largest for shear insensitive cracks. The remaining curves are for the criteria in Eqs. (26), (27), (31), and (36) respectively. We note that the ratio $k_{eq\ bias}/k_{uniax}$ increases with m for all criteria, and is smallest for the fracture criteria based on strain energy release rate. Also included in the comparison is the assumption that with respect to fracture, the principal stresses behave independently. This leads to a constant ratio of 2. The comparison suggests that for $m \lesssim 3$ the independence assumption will lead to unconservative estimates of the statistics of failure in biaxial tension. We note in passing that for the low values of P_f desired in most structural applications, the failure probability ratio is the same as the k ratio.

Experimental data on uniaxial and equibiaxial bending of alumina plates were recently obtained by Sines and Giovan [30]. In Fig. 4 the equibiaxial data are approximated by a Weibull 2-parameter curve. The uniaxial results were computed for shear-insensitive cracks and also for the fracture criteria listed in Eq. (27) and Eq. (36). As would be anticipated on the basis of the preceding discussion, the agreement is best using Eq. (36). For details of the analysis, see [30].

The relative merits of various fracture criteria can also be tested in other ways. For instance, Petrovic and Mendiratta [32] tested the variation of tensile fracture stress of controlled surface cracks with crack angle. The conclusion from this experiment is that Eqs. (31) and (36) are in reasonably good agreement with experiment, and are the best of the criteria listed in this paper.

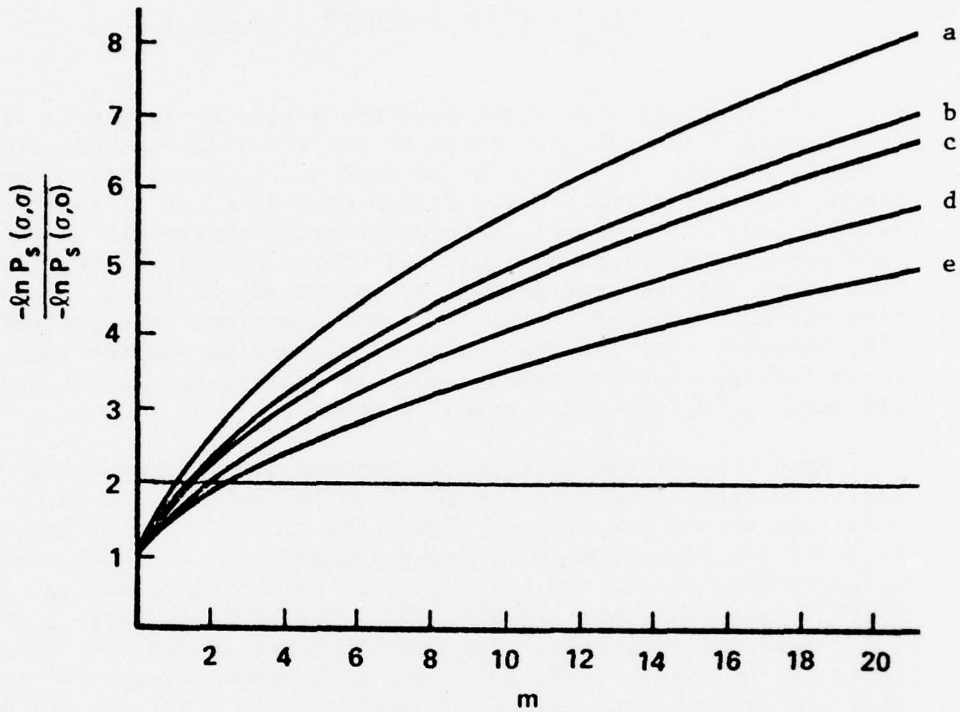

Fig. 3. Relation Between Failure Probability Under Equibiaxial Loading and That for Uniaxial Loading as a Function of Weibull Parameter m.

WLT FOR ANISOTROPIC MATERIALS

Anisotropic brittle materials may be anisotropic with respect to their elastic properties, their fracture properties, or both. The elastic anisotropy will influence the stresses resulting from a given load application, but will not directly affect the calculation of probability of failure, which depends only on the stress state and crack distribution.

Two different approximate techniques have been proposed for determining the fracture statistics of certain types of polygraphite that are isotropic in one plane but have a lower fracture stress in the direction normal to the plane of isotropy. In one approach, the cracks were assumed randomly oriented, but they were given critical stresses that varied with orientation [31]. In the other, the critical stresses were assumed unaffected by orientation, but cracks were given a preferred orientation [32]. It is likely that there is actually both a preferred orientation and a variation

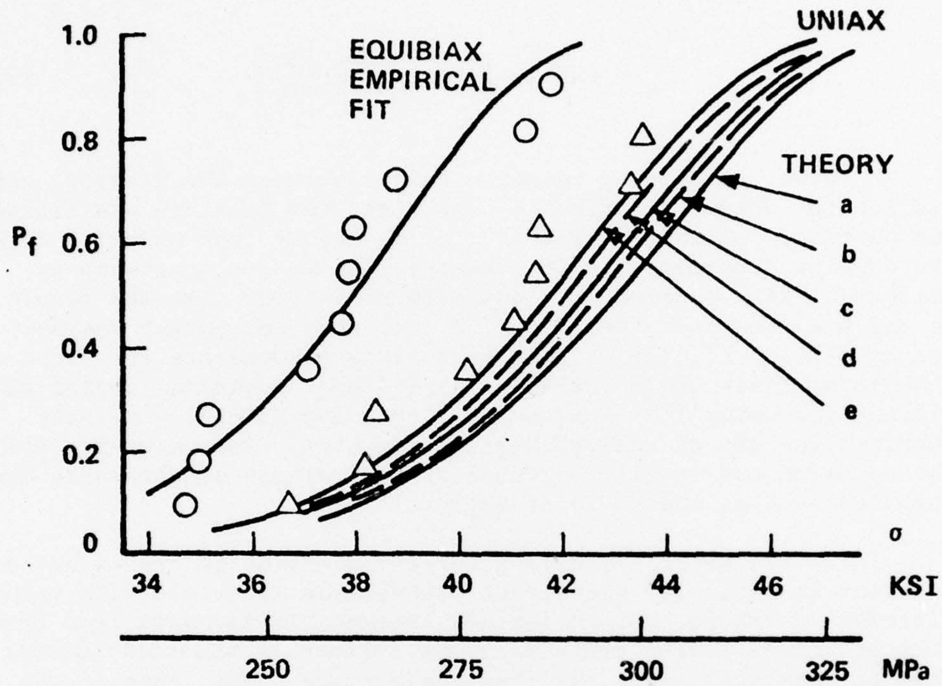

Fig. 4. Theory Compared with Data of Giovan and Sines

in crack strength with orientation, but a theory incorporating both would be considerably more complex and not really needed, at least for graphite. Both theories that attribute the entire anisotropy to only one effect appear to be in satisfactory agreement with the available data.

WLT FOR SURFACE DISTRIBUTED CRACKS

In his first paper on the statistics of fracture, Weibull included a discussion of surface-distributed flaws [1]. In this treatment, the only change was to replace $B = \int n_1(\sigma)\, dV$ by $B = \int n_2(\sigma)\, dA$ where A represents area, and n_2 might or might not be the same as n_1. Thus the analogy to the theory for volume distribution flaws is very close.

In a more recent analysis the fracture statistics of surface distributed cracks have been worked out on the assumption that crack planes are always normal to the material surface [35]. As a result, the orientation of a crack is specified by a single angle, whereas for volume distributed cracks two angles are required. The analogue to Eq. (8) applicable to surface distributed cracks of the type under discussion is

$$P_s = \exp\left[-\iint dA\, d\sigma_c \frac{\omega}{\pi} \frac{dN}{d\sigma_c}\right] \tag{37}$$

A rather surprising relation exists between the fracture statistics for volume distributed cracks and the fracture statistics for surface distributed cracks. Let us assume that uniaxial data are used to determine the fracture statistics for specimens in uniform biaxial tension. It has been shown [36] that the result is the same for both theories, i.e., it does not matter whether we use Eq. (8) or Eq. (37). This is quite a convenience for those who like to do their own calculating. Finding the biaxial stress statistics for Weibull's 3-parameter form using Eq. (8) generally requires the use of a large digital computer. But using Eq. (37) the solution can readily be found using a simple programmable hand computer such as the HP-25 or SR-56.

There are materials having surface distributed cracks but no interior cracks, e.g. glass, but probably no materials with volume distributed cracks, but no surface cracks. It is known from fracture mechanics that a crack near the surface is generally weaker than an identical crack far from the surface. This complicates the interpretation of laboratory data on small specimens and the application of the results to larger specimens. It becomes necessary to separate out the surface and volume contributions to the fracture statistics. An analysis of this type has recently been carried out by Rebholz and Teter [37].

MICROSTRUCTURAL CONSIDERATIONS

So far our discussion of fracture statistics has been based on a combination of extreme value theory and fracture mechanics. The crack density function had to be inferred from test data, and it was pointed out that extrapolation of this function to stresses outside the region bounded by the highest and lowest observed fracture stresses is unwarranted. Thus without additional sources of information it is not possible to obtain reliable estimates of the fracture behavior of very much larger specimens or the stresses at which extremely low probabilities of failure can be assured.

To resolve this difficulty, McClintock [23] recently proposed a theory for the crack density function based on microstructural considerations. He assumed that cracks are random aggregations of imperfectly bonded pairs of grains. If the probability that two adjacent grains are unbonded is q, the probability that two such unbonded pairs are side by side is q^2, etc. In this manner the statistics of crack size can be found, and using the value of the critical stress intensity factor for the material this can be converted into statistics of critical stress. From this in turn one can obtain the statistics of failure. The material model employed by McClintock was a 2-dimensional one in which all grains were identical rectangles and all cracks were normal to the applied tensile stress. It was shown that the probability of a crack being longer than a is exp $[-a/\lambda]$, and that the resulting fracture statistics do not approach an asymptotic form for large specimens.

This theory was similar in many ways to one published earlier by Fisher and Holloman [38]. These authors considered randomly oriented, penny-shaped cracks in a 3-dimensional elastic solid. They made the judgmentally-based assumption that the probability of a crack having a radius larger than r is exp $[-r/\lambda]$. Using this size distribution they used fracture mechanics to obtain a critical stress distribution. Employing the assumption $\sigma_e = \sigma_n$, they then worked out the fracture statistics. The relation of their work to Weibull's and other WLT's escaped general attention because they retained the binomial form in their calculations rather than converting to exponentials [see Eq. (1)].

Batdorf [39] applied the McClintock concept of the origin of cracks to a consideration of randomly oriented penny-shaped cracks in a 3-dimensional polycrystal. He found that the probability of a crack having a radius greater than r is exp $[-r^2/\lambda^2]$. The fracture criterion employed was that expressed in Eq. (31).

Among the principal findings of this theory are the following:

(1) The total number of cracks is finite rather than infinite as in Weibull theory. The most probable critical stress is $s_c \approx 1$ where s is a reduced stress to be discussed later. Weibull's 2- and 3-parameter forms imply that the number of cracks with a given fracture stress increases monotonically with stress; thus the most probable fracture stress is infinity.

(2) The probability of failure in simple tension is given by

$$P_f(s) = 1 - \exp\left[-N_oV \int_o^s \left(1 - \frac{s_c}{s}\right) \frac{4}{s_c^5} \exp\left(-1/s_c^4\right) d\, s_c\right] \quad (38)$$

where N_oV is the total number of cracks. For a sufficiently small probability of failure

$$P_f(s) \cong (NV_o/4)s^4 \exp(-1/s^4) \qquad (39)$$

(3) Examining Eq. (39) we see that $P_f(s) = 0$ for $s = 0$, and $P_f(s) > 0$ when $s > 0$. However, $P_f(s)$ cannot be expressed as a power series in s because at $s = 0$, P_f and all its derivatives vanish. It turns out that over a finite range of P_f such as $0.01 \leq P_f \leq 0.99$, the prediction of this theory can be fitted very closely using Weibull's 3-parameter form. However, m and σ_u are both very weak functions of the volume instead of being volume independent as in Weibull theory. There is some experimental confirmation of the predicted direction of change of Weibull parameters with volume change [39].

(4) The reduced stress is related to the actual stress through the equation

$$s = 2\sigma A_o^{0.25} / K_{Ic}\, \pi^3 \left[\ell n\, (q^{-1})\right]^{0.25} \qquad (40)$$

where

A_o is grain cross sectional area

K_{Ic} is the critical stress infinity factor for mode I

q is the probability that adjacent grains will be unbonded.

Knowledge of how the fracture statistics depend on grain size, effectiveness of intergranular bonding, and critical stress intensity factor should be very helpful in determining what changes in processing variables will effect desired improvements in material properties. Results qualitatively similar to those just discussed are obtained with McClintock's theory and that of Fisher and Holloman, except that the latter considered only a structureless elastic solid, and therefore contained no analogue to Eq. (40).

More recently McClintock has refined his 2-dimensional model by using regular hexagons for grains and assuming grain boundaries have a Weibull-type strength distribution [40]. Such a model assumes no cracks are present initially, but as the load increases, cracks are created, grow, coalesce, and eventually cause the load capacity of the specimen to pass its peak and decline.

Another theory in which cracks are created by the loading process has been devised for polygraphites [41]. This theory is based on a consideration of the low tensile strength of a graphite crystal in the c-direction. The grains are assumed to be randomly oriented, so that here and there a chance aggregation of grains of nearly the same orientation are arrayed in the shape of a penny. Such an array opens up and acts like a crack when the applied stress exceeds the capacity of all of its constituent grains.

Two-dimensional models are generally of qualitative rather than quantitative value, but they help guide our thinking. Three-dimensional models are obviously preferable in principle, but their complexity requires greater development time and effort. At present, simplifying assumptions are used whose influence on the accuracy of predictions is somewhat uncertain. However, statistical theories of fracture incorporating extreme value theory, fracture mechanics and microstructural considerations are as yet in their infancy. The present writer believes that such theories offer the greatest long range promise for future progress.

STATIC FATIGUE

Up to this point we have discussed primarily idealized short term brittle fracture in which preexisting cracks are unaffected by increasing stress until their strength is exceeded, at which time they expand suddenly and fracture the specimen or structural part. Some attention was devoted to situations in which cracks are created and grow as the stress increases. We now consider fracture resulting from subcritical crack growth — slow growth of cracks with passage of time, even at constant stress. Whereas the objective in short-term fracture is to predict the probability of failure in a given stress state, in static fatigue it is to predict the time to reach a given probability of failure when the stress state is specified.

It has been determined by a number of investigators conducting mode I crack propagation studies that for a given system [material, temperature, environment, etc.] there is a unique relationship between crack velocity and the crack tip intensity factor K_I [42,43]. This is usually expressed in the form

$$v = A K_I^n \tag{41}$$

If a is the crack length

$$v = da/dt \tag{42}$$

and

$$K_I = \sigma Y \sqrt{a} \tag{43}$$

where Y is a geometrical factor. Combining these relations with Eq. (41) it is readily shown [44] that the time to failure is given by

$$T = 2\left[K_{Ii}^{2-n} - K_{Ic}^{2-n}\right] \left[(n - 2)\, \sigma^2\, AY^2\right] \qquad (44)$$

where K_{Ii} is the initial value of K_I.

The problem now is how to go from a laboratory situation with an artificial crack of prescribed size growing in mode I to the general situation in service, in which an unknown distribution of randomly oriented natural cracks undergoes mixed mode growth in an arbitrary stress state. Such a goal has yet to be achieved. In fact it is beyond the state of the art in fracture mechanics, since it involves a knowledge of the growth rate and eventual critical stress of a non-planar crack. However, significant progress is being made.

Most theoretical work in this area depends on three basic assumptions or limitations (1) the stress state is uniform simple tension, (2) crack planes are assumed to be normal to the applied stress both for calculating crack size and for determining crack velocity, (3) the short-term fracture statistics are adequately described by Weibull's 2-parameter form.

Davidge, McLarin, and Tappin [44] have used these assumptions to develop strength-probability-time (SPT) relations. An SPT diagram for alumina is shown in Fig. 5. Such a diagram can be used to find the stress corresponding to an acceptable probability of failure during the design life of a structural element.

Evans and Wiederhorn [45] have shown how the statistics of failure are affected by prior proof testing. The minimum time to failure is found by noting that no crack longer than the critical length for the proof test can be present in any of the surviving specimens. They then use Weibull statistics and Eq. (43) to obtain the actual distribution of crack size in the surviving specimens. From this they solve for the probability of fracture as a function of stress and time under load. Results for soda-lime glass in water are shown in Fig. 6. The comparison between theory and experiment suggests that the former may be conservative by between a half and a whole order of magnitude. Discrepancies of this sort may be due in part to errors in determining material parameters and propagation constants. The manner in which these affect the statistics of fracture in static fatigue have recently been analyzed in some detail [46,47].

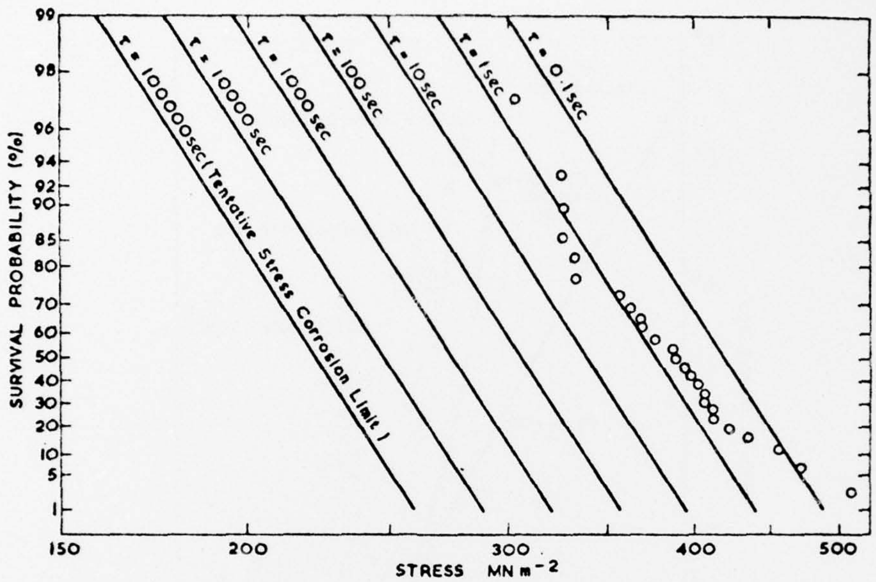

Fig. 5. SPT Diagram for Alumina [44]

DYNAMIC FRACTURE

In dynamic fracture studies, an intense stress pulse of very short duration passes through the material. Many cracks grow. After passage of the stress wave, the specimen may still be in one piece but in damaged condition, or it may be fragmented into a few or many pieces.

In treating internal crack damage, Curran, Shockey and Seaman [47,48] have developed a fracture model that takes into account three aspects of the fracture process not usually included in treatments of short-term fracture: (a) nucleation of cracks as a function of stress; (b) growth of cracks as a function of stress, time, fracture toughness, and initial crack size; (c) decrease in strength and stress attenuation with increasing damage.

The nucleation rate is assumed to have the form

$$\dot{N} = N_o \exp\left[\left(\sigma - \sigma_{no}\right)/\sigma_1\right] \tag{45}$$

where σ_{no} is the threshold stress for nucleation, and N_o and σ_1 are constants. The cracks are assumed to be nucleated with a distribution of radii given by

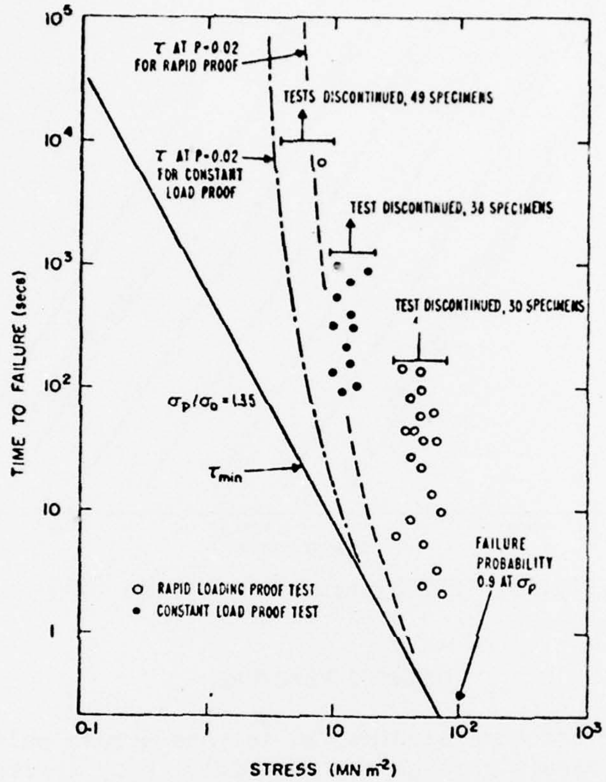

Fig. 6. A Comparison of Failure Times After Proof Testing [45]

$$\Delta N(r) = \Delta N_o \exp\left[- r/r_1\right] \tag{46}$$

where ΔN_o is the total number/cm^3 nucleated in the time interval t to t + Δt, ΔN is the number having a radius greater than r, and r_1 is a constant. The growth of cracks was tested in many ways, the most successful being the assumption that

$$\dot{r} = r\,(\sigma - \sigma_{go})/4\eta \tag{47}$$

where η is the crack tip viscosity. The threshold stress for crack growth is assumed to be given by

$$\sigma_{go} = 0.5\, K_{Ic} \sqrt{\pi/r} \tag{48}$$

where K_{Ic} is the plane strain fracture toughness.

As the stress wave progresses through a material, energy is extracted in creating and enlarging cracks, so the pulse is attenuated. The calculation of the number and size of cracks as a function of time and location, their effects on the stiffness and damping of the material, and the effects of these in turn on the stress pulse propagation is a strongly coupled and involved process requiring a computer code. Accordingly we must expect to be limited to numerical results for the particular cases chosen for investigation. However the number and size distributions of cracks in damaged specimens and fragment size in fragmented specimens are in reasonable agreement with experiment.

In the case of extremely short stress pulses the diffraction of the wave by cracks and delay times in the initiation of crack growth become important. These and other refinements in the theory have been discussed by Kalphoff and Seaman [50] and by Vardar and Finnie [51].

CONCLUDING DISCUSSION

It appears from the foregoing that Weibull's theory for uniaxial stress states is essentially correct as it stands, except for the limitations that it applies only to tensile type fractures, and that caution must be exercised in making predictions implying a knowledge of $n(\sigma)$ outside of the stress range in which it has been established by experiment. We have shown that Weibull's treatment of polyaxial stress states implies the assumption that cracks are shear insensitive, i.e., $\sigma_e = \sigma_n$. Eqs. (8) and (11) allow for the use of arbitrarily chosen fracture criteria in analyzing polyaxial stress states. Present evidence suggests that Eq.(31) is somewhere near right, and leads to better results than either Weibull theory or the assumption of independence of principal stresses. More research is needed, however, in the areas of crack interaction and shear-type fractures, where weakest link theory does not apply.

Theories including due consideration of extreme value theory, fracture mechanics and material microstructure have only recently been introduced into the literature. Although complex, they offer the greatest long range promise and should be developed much more completely.

Much progress has recently been made in the statistical treatment of static fatigue, including the effects of proof testing. The accuracy of predictions is impaired somewhat, however, by the tacit assumption that the critical crack is normal to the applied stress. Also little has been done to analyze time to failure under polyaxial stress conditions.

In dynamic fracture, considerable success has been achieved in accounting for experimental data on crack damage and fragmentation in the case of materials that have been studied fairly thoroughly. There are so many material parameters, however, that optimizing their values to match theory to experiments involving all of them simultaneously may not result in reliable values for each parameter. This in turn makes transfer of the knowledge gained to untested materials difficult.

We conclude from all this that a lot of progress has been made in statistical theory of fracture, and that useful techniques exist to guide designers in their consideration of short-term, long duration, and dynamic fracture conditions. It is also evident that a lot of work remains to be done.

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